

ON (p, r) -NULL SEQUENCES AND THEIR RELATIVES

KATI AIN AND EVE OJA

Dedicated to Professor Albrecht Pietsch on his eightieth birthday

ABSTRACT. Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$, where p^* is the conjugate index of p . We prove an omnibus theorem, which provides numerous equivalences for a sequence (x_n) in a Banach space X to be a (p, r) -null sequence. One of them is that (x_n) is (p, r) -null if and only if (x_n) is null and relatively (p, r) -compact. This equivalence is known in the “limit” case when $r = p^*$, the case of the p -null sequence and p -compactness. Our approach is more direct and easier than those applied for the proof of the latter result. We apply it also to characterize the unconditional and weak versions of (p, r) -null sequences.

1. INTRODUCTION

Let X be a Banach space and let $c_0(X)$ denote the space of null sequences in X . Recently, Delgado and Piñeiro [24] introduced and studied an interesting class of p -null sequences, where $p \geq 1$, which is a linear subspace of $c_0(X)$. In [21], it was proved that the space of p -null sequences in X can be identified with the Chevet–Saphar tensor product $c_0 \hat{\otimes}_{d_p} X$.

On the other hand, there is a strong form of compactness, the p -compactness, that has been studied during the last dozen years in the literature (see, e.g., [1, 3, 9, 11, 12, 18, 23, 27]). The p -null sequences can be characterized via the p -compactness as follows. (The definitions will be given in Section 2.)

Theorem 1.1 (Delgado–Piñeiro–Oja). *Let $1 \leq p < \infty$. A sequence (x_n) in a Banach space X is p -null if and only if (x_n) is null and relatively p -compact.*

Theorem 1.1 was discovered in [24, Proposition 2.6] and proved in the case of Banach spaces enjoying a version of the approximation property depending on p (by [20], this version of the approximation property coincides with the classical one for the closed subspaces of $L_p(\mu)$ -spaces). For arbitrary Banach spaces, Theorem 1.1 was proved in [21].

The proof of Theorem 1.1 in [21] relies on the above-mentioned description of the space of p -null sequences as a Chevet–Saphar tensor product. Very recently, an alternative natural proof was found by Lassalle and Turco [19] who rediscovered and applied a powerful theory due to Carl and Stephani [7]

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from 1984. Key concepts of the Carl–Stephani theory are \mathcal{A} -null sequences and \mathcal{A} -compact sets in Banach spaces, which are defined for an arbitrary operator ideal \mathcal{A} . Lassalle–Turco’s proof in [19] relies on the following operator ideal version of Theorem 1.1, deduced from the Carl–Stephani theory in [19, Proposition 1.4].

Theorem 1.2 (Lassalle–Turco). *Let \mathcal{A} be an operator ideal. A sequence (x_n) in a Banach space X is \mathcal{A} -null if and only if (x_n) is null and \mathcal{A} -compact.*

A starting point for the present article was the observation that, in the proof of Theorem 1.1, Theorem 1.2 could be used in a more efficient way than in [19]. In particular, the technical result [19, Proposition 1.5] would not be needed in the proof. Even more, it is obtained for “free” as a by-product (see Remark 3.2). Moreover, in that way, Theorem 1.2 can be applied to prove results similar to Theorem 1.1 also in cases when the method of [21] cannot be applied. One of such cases is, for instance, the one that involves the recent concepts of (p, r) -compactness [1] and of (p, r) -null sequences [2].

In Section 3, we prove an omnibus theorem, Theorem 3.1, which provides six equivalent properties for a sequence in a Banach space to be a (p, r) -null sequence. For completeness, let us cite here the part of the omnibus Theorem 3.1 which directly corresponds to Theorem 1.1.

Theorem 1.3. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$, where p^* denotes the conjugate index of p . A sequence (x_n) in a Banach space X is (p, r) -null if and only if (x_n) is null and relatively (p, r) -compact.*

Let us remark that in the “limit” case $r = p^*$, the (p, p^*) -null and (p, p^*) -compactness are precisely the p -null and p -compactness. This is, in fact, the only special case when Theorem 1.3 could be proved by the method in [21]. The reason is simple: the method in [21] uses the Hahn–Banach theorem. But the (p, r) -context provides a suitable norm only if $r = p^*$, and in all other cases merely quasi-norms are available. But, as well known, quasi-normed spaces do not enjoy the Hahn–Banach theorem.

The approach developed in Section 3 is applied in Section 4 to characterize the unconditional and weak versions of (p, r) -null sequences.

Our notation is standard. We consider Banach spaces over the same, either real or complex, field \mathbb{K} . The closed unit ball of a Banach space X is denoted by B_X .

We denote by \mathcal{L} , \mathcal{W} , \mathcal{K} , and $\overline{\mathcal{F}}$, respectively, the operator ideals of bounded, weakly compact, compact, and approximable linear operators. We refer to Pietsch’s book [22] and the survey paper [14] by Diestel, Jarchow, and Pietsch for the theory of operator ideals. Let us recall here only the definition of the operator ideal \mathcal{A}^{sur} , the *surjective hull* of an operator ideal \mathcal{A} (see [30, Section 2] and [22, 4.7.1]). An operator $T \in \mathcal{L}(Y, X)$ belongs to $\mathcal{A}^{\text{sur}}(Y, X)$ if $Tq \in \mathcal{A}(Z, X)$ for some surjection $q \in \mathcal{L}(Z, Y)$. Obviously, $\mathcal{A} \subset \mathcal{A}^{\text{sur}}$. If $\mathcal{A} = \mathcal{A}^{\text{sur}}$, then \mathcal{A} is called *surjective*.

The Banach space of all absolutely p -summable sequences in X is denoted by $\ell_p(X)$ and its norm by $\|\cdot\|_p$. By $\ell_p^w(X)$ we mean the Banach space of weakly p -summable sequences in X with the norm $\|\cdot\|_p^w$ (see, e.g., [15, pp. 32–33]). If $1 \leq p \leq \infty$, then p^* denotes the conjugate index of p (i.e., $1/p + 1/p^* = 1$ with the convention $1/\infty = 0$).

To simplify notation, we shall use the symbol ℓ_∞ instead of c_0 and, more generally, $\ell_\infty(X)$ instead of $c_0(X)$ if X is a Banach space.

2. BASIC CONCEPTS AND NOTATION

2.1. The (p, r) -compactness of sets and operators. Let X be a Banach space. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We define the (p, r) -convex hull of a sequence $(x_k) \in \ell_p(X)$ by

$$(p, r)\text{-conv}(x_k) = \left\{ \sum_{k=1}^{\infty} a_k x_k : (a_k) \in B_{\ell_r} \right\}.$$

As in [1], we say that a subset K of X is *relatively (p, r) -compact* if $K \subset (p, r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p(X)$. According to Grothendieck's criterion, the $(\infty, 1)$ -compactness coincides with the usual compactness (because $(\infty, 1)\text{-conv}(x_n)$ is precisely the closed absolutely convex hull of (x_n)). The $(p, 1)$ -compactness was occasionally considered in the 1980s by Reinov [25] and by Bourgain and Reinov [6] in the study of approximation properties of order $s \leq 1$. The (p, p^*) -compactness was introduced in 2002 by Sinha and Karn [27] under the name of *p -compactness*. Remark that the 1-compactness was considered already in 1973 by Stephani [30, Section 4] under the name of nuclearity (of sets) (see also Remark 2.3).

The notion of p -null sequences is due to Delgado and Piñeiro [24]. It was extended in [2] in a verbatim way as follows. We call a sequence (x_n) in X (p, r) -null if for every $\varepsilon > 0$ there exist $(z_k) \in \varepsilon B_{\ell_p(X)}$ and $N \in \mathbb{N}$ such that $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$. The p -null sequences in [24] are precisely the (p, p^*) -null sequences.

A useful way to look at (p, r) -convex hulls is the following. It is well known and easy to see that every $(x_k) \in \ell_p(X)$ defines a compact, even approximable, operator $\Phi_{(x_k)} : \ell_r \rightarrow X$ through the equality

$$\Phi_{(x_k)}(a_k) = \sum_{k=1}^{\infty} a_k x_k, \quad (a_k) \in \ell_r.$$

Clearly,

$$(p, r)\text{-conv}(x_k) = \Phi_{(x_k)}(B_{\ell_r}).$$

In [1], (p, r) -compact operators were introduced in an obvious way: a linear operator $T : Y \rightarrow X$ is (p, r) -compact if $T(B_Y)$ is a relatively (p, r) -compact subset of X . Let $\mathcal{K}_{(p, r)}$ denote the class of all (p, r) -compact operators acting between arbitrary Banach spaces. Then $\mathcal{K}_{(p, p^*)} = \mathcal{K}_p$, the class of *p -compact operators in the sense of Sinha–Karn* [27]. And $\mathcal{K}_{(p, 1)}$ is the class of *p -compact operators in the Bourgain–Reinov sense* (cf. [6, 25]).

Properties of \mathcal{K}_p were studied in [27] and, for instance, in the recent papers [12, 13, 28]. In [1], an alternative approach, which is direct and easier than in these articles, was developed to study the (quasi-Banach) operator ideal structure of $\mathcal{K}_{(p, r)}$, among others, encompassing and clarifying main results on $\mathcal{K}_p = \mathcal{K}_{(p, p^*)}$. (Remark that in the latter case the same approach was independently developed by Pietsch [23] yielding an important far-reaching theory of the (Banach) operator ideal \mathcal{K}_p .)

The approach in [1] starts as follows. One observes that $\mathcal{K}_{(p,r)}$ is a surjective operator ideal (an easy straightforward verification). Another immediate observation is that

$$\Phi_{(x_n)} \in \mathcal{N}_{(p,1,r^*)}(\ell_r, X),$$

the space of $(p, 1, r^*)$ -nuclear operators (for the definition of $\mathcal{N}_{(t,u,v)}$, see [22, 18.1.1]). But then, by the definition of the surjective hull, the injective associate of $\Phi_{(x_n)}$ belongs to $\mathcal{N}_{(p,1,r^*)}^{\text{sur}}$. Let us denote it by $\overline{\Phi}_{(x_n)}$. Observing that any $T \in \mathcal{K}_{(p,r)}(Y, X)$ can be factorized as $T = \overline{\Phi}_{(x_n)}S$, one easily obtains that

$$\mathcal{K}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}^{\text{sur}}$$

as operator ideals (see [1, Theorem 3.2]).

2.2. Some classes of bounded sets. Let us introduce some useful notation which is inspired by [31], but seems to be more suggestive than the notation in [31].

Let \mathbf{b} denote the class of all bounded subsets of all Banach spaces, and let \mathbf{g} be a subclass of \mathbf{b} . Let X be a Banach space. Following [31, Definition 1.1], we denote by $\mathbf{g}(X)$ the family of subsets of X which are of type \mathbf{g} . For instance, $\mathbf{b}(X)$ is the family of all bounded subsets of X .

We denote by \mathbf{w} and \mathbf{k} , respectively, the classes of all relatively weakly compact and relatively compact subsets of all Banach spaces. It is convenient to denote by $\mathbf{k}_{(p,r)}$ the class of all relatively (p, r) -compact sets in all Banach spaces. In particular, $\mathbf{k} = \mathbf{k}_{(\infty,1)}$ and $\mathbf{k}_p := \mathbf{k}_{(p,p^*)}$, the class of all relatively p -compact sets.

Let \mathcal{A} be an operator ideal. Denote by $\mathcal{A}(\mathbf{g})$ the subclass of \mathbf{b} , which is given as

$$\mathcal{A}(\mathbf{g})(X) = \{E \subset X : E \subset T(F) \text{ for some } F \in \mathbf{g}(Y) \text{ and } T \in \mathcal{A}(Y, X)\}$$

where X is an arbitrary Banach space (in [31], the notation $\mathcal{A} \circ \mathbf{g}$ is used).

In this notation, Grothendieck's criterion of compactness reads as follows.

Proposition 2.1 (Grothendieck). *One has $\mathbf{k} = \overline{\mathcal{F}}(\mathbf{b}) = \mathcal{K}(\mathbf{b})$.*

Proof. Let X be a Banach space and let $K \in \mathbf{k}(X)$. Grothendieck's criterion gives us a sequence $(x_n) \in c_0(X)$ such that $K \subset \Phi_{(x_n)}(B_{\ell_1})$. Since $\Phi_{(x_n)} \in \overline{\mathcal{F}}(\ell_1, X)$, it is clear that K is of type $\overline{\mathcal{F}}(\mathbf{b})$. But $\overline{\mathcal{F}}(\mathbf{b}) \subset \mathcal{K}(\mathbf{b})$ because $\overline{\mathcal{F}} \subset \mathcal{K}$. Finally, if K is of type $\mathcal{K}(\mathbf{b})$, then it is relatively compact. \square

Proposition 2.1 says, in particular, that $\mathbf{k}_{(\infty,1)} = \mathcal{K}_{(\infty,1)}(\mathbf{b})$. Using the definitions of $\mathbf{k}_{(p,r)}$ and $\mathcal{K}_{(p,r)}$ together with the observation (see Section 2.1) that $\Phi_{(x_n)}$ belongs to the operator ideal $\mathcal{N}_{(p,1,r^*)}$, the above proof yields also the general case.

Proposition 2.2. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then $\mathbf{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\mathbf{b}) = \mathcal{K}_{(p,r)}(\mathbf{b})$.*

Remark 2.3. Using the notion of ideal system of sets (see [30]), the equalities $\mathbf{k} = \mathcal{K}(\mathbf{b})$ and $\mathbf{w} = \mathcal{W}(\mathbf{b})$ were observed in [31]. In the special case $p = 1, r = \infty$, the left-hand equality $\mathbf{k}_1 = \mathbf{k}_{(1,\infty)} = \mathcal{N}(\mathbf{b})$ of Proposition 2.2

was proved in [30]; here $\mathcal{N} = \mathcal{N}_{(1,1,1)}$ denotes, as usual, the operator ideal of (classical) nuclear operators.

2.3. \mathcal{A} -null sequences and \mathcal{A} -compact sets. Let us now describe the relevant notions (cf. Theorem 1.2) from the Carl–Stephani theory [7], which is based on earlier work by Stephani [29–31].

Let \mathcal{A} be an operator ideal.

Following [7, Lemma 1.2], a sequence (x_n) in a Banach space X is called \mathcal{A} -null if there exist a Banach space Y , a null sequence (y_n) in Y , and $T \in \mathcal{A}(Y, X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$.

Using the notation of Section 2.2 and following [7, Theorem 1.2], we say (as in [19]) that a subset K of a Banach space X is \mathcal{A} -compact if K is of type $\mathcal{A}(\mathbf{k})$, i.e. $K \in \mathcal{A}(\mathbf{k})(X)$.

Using Proposition 2.1 and 2.2 we shall see now that the relatively (p, r) -compact sets, $\mathcal{N}_{(p,1,r^*)}$ -compact sets, and $\mathcal{K}_{(p,r)}$ -compact sets are all the same.

Proposition 2.4. *Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. Then $\mathbf{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\mathbf{k}) = \mathcal{K}_{(p,r)}(\mathbf{k})$.*

Proof. We know that $\mathcal{N}_{(p,1,r^*)}$ is a minimal operator ideal (see [22, 18.1.4]). This means that $\mathcal{N}_{(p,1,r^*)} = \overline{\mathcal{F}} \circ \mathcal{N}_{(p,1,r^*)} \circ \overline{\mathcal{F}}$ (see [22, 4.8.6]). Hence, using Propositions 2.2 and 2.1, we have

$$\begin{aligned} \mathcal{K}_{(p,r)}(\mathbf{k}) &\subset \mathcal{K}_{(p,r)}(\mathbf{b}) = \mathbf{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\mathbf{b}) = (\overline{\mathcal{F}} \circ \mathcal{N}_{(p,1,r^*)})(\overline{\mathcal{F}}(\mathbf{b})) \\ &= \overline{\mathcal{F}} \circ \mathcal{N}_{(p,1,r^*)}(\mathbf{k}) \subset \mathcal{N}_{(p,1,r^*)}(\mathbf{k}) \subset \mathcal{K}_{(p,r)}(\mathbf{k}). \end{aligned}$$

This shows that $\mathbf{k}_{(p,r)} = \mathcal{N}_{(p,1,r^*)}(\mathbf{k}) = \mathcal{K}_{(p,r)}(\mathbf{k})$. \square

Remark 2.5. The second equality in Proposition 2.4 also follows from the general Carl–Stephani theory. Indeed, for any operator ideal \mathcal{A} , it is known (see [7, p. 79]) that a subset is \mathcal{A} -compact if and only if it is \mathcal{A}^{sur} -compact. And (see Section 2.1) $\mathcal{N}_{(p,1,r^*)}^{\text{sur}} = \mathcal{K}_{(p,r)}$.

3. AN OMNIBUS CHARACTERIZATION OF (p, r) -NULL SEQUENCES

Theorem 3.1 below is an omnibus theorem, which provides six equivalent properties for a sequence in a Banach space to be a (p, r) -null sequence. One of these properties is to be a uniformly (p, r) -null sequence, which is a natural (formal) strengthening of a (p, r) -null sequence.

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. We call a sequence (x_n) in a Banach space X *uniformly (p, r) -null* if there exists $(z_k) \in B_{\ell_p(X)}$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in \varepsilon(p, r)\text{-conv}(z_k)$ for all $n \geq N$.

We say that (x_n) is *uniformly p -null* if it is uniformly (p, p^*) -null. The latter property was implicitly used in a result by Lassalle and Turco asserting (in the above terminology) that the p -null sequences are always uniformly p -null (concerning the proof (and its simple alternative), see Remark 3.2).

Theorem 3.1. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (a) (x_n) is (p, r) -null,
- (b) (x_n) is null and relatively (p, r) -compact,
- (c) (x_n) is null and $\mathcal{N}_{(p,1,r^*)}$ -compact,
- (d) (x_n) is null and $\mathcal{K}_{(p,r)}$ -compact,
- (e) (x_n) is $\mathcal{N}_{(p,1,r^*)}$ -null,
- (f) (x_n) is $\mathcal{K}_{(p,r)}$ -null,
- (g) (x_n) is uniformly (p, r) -null.

Proof. An easy verification of (a) \Rightarrow (b) can be found in [2, Proposition 2]. For completeness and easy reference, let us present it here.

Since (x_n) is (p, r) -null, for every $\varepsilon > 0$ there are $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$, $\|(z_k)\|_p \leq \varepsilon$, such that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_r}$, for all $n \geq N$. Hence, for all $n \geq N$,

$$\|x_n\| \leq \sum_{k=1}^{\infty} \|a_k^n z_k\| \leq \|(a_k^n)_k\|_{p^*} \|(z_k)\|_p \leq \|(a_k^n)_k\|_r \|(z_k)\|_p \leq \varepsilon,$$

and therefore $x_n \rightarrow 0$.

Since $\{x_N, x_{N+1}, \dots\} \subset (p, r)$ -conv (z_k) and $(z_k) \in \ell_p(X)$, the sequence

$$y_k = \begin{cases} x_k & \text{if } k < N, \\ z_{k-N+1} & \text{if } k \geq N, \end{cases}$$

is in $\ell_p(X)$ and $x_n \in (p, r)$ -conv (y_k) for all $n \in \mathbb{N}$. This means that (x_n) is relatively (p, r) -compact.

Implications (b) \Leftrightarrow (c) \Leftrightarrow (d) are immediate from Proposition 2.4.

Implications (c) \Leftrightarrow (e) and (d) \Leftrightarrow (f) are immediate from Theorem 1.2.

To prove that (f) \Rightarrow (g), let (x_n) be a $\mathcal{K}_{(p,r)}$ -null sequence. Then there are a null sequence (y_n) in a Banach space Y and an operator $T \in \mathcal{K}_{(p,r)}(Y, X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$. The (p, r) -compactness of T gives us a sequence $(w_k) \in \ell_p(X)$ such that $T(B_Y) \subset (p, r)$ -conv (w_k) . Now $(z_k) := \left(\frac{w_k}{\|(w_k)\|_p}\right) \in B_{\ell_p(X)}$, and let $\varepsilon > 0$. As (y_n) is null in Y , for $\varepsilon_0 := \frac{\varepsilon}{\|(w_k)\|_p}$ there exists $N \in \mathbb{N}$ such that $Ty_n \in \varepsilon_0 T(B_Y)$ for all $n \geq N$. Hence,

$$x_n \in \varepsilon_0 (p, r)\text{-conv}(w_k) = \varepsilon_0 \|(w_k)\|_p (p, r)\text{-conv}(z_k) = \varepsilon (p, r)\text{-conv}(z_k)$$

for all $n \geq N$, as desired.

The implication (g) \Rightarrow (a) is clear from the definitions, because if $(z_k) \in B_{\ell_p(X)}$, then $(\varepsilon z_k) \in \varepsilon B_{\ell_p(X)}$ and (p, r) -conv $(\varepsilon z_k) = \varepsilon (p, r)$ -conv (z_k) . \square

Remark 3.2. In the special case when $r = p^*$, Theorem 3.1 contains Theorem 1.1, complementing it and providing for it a somewhat easier proof than in [19]. In fact, the technical Lassalle–Turco result [19, Proposition 1.5] (inspired by [3, Theorem 1]) is not needed. Even more, this technical result appears as a simple by-product of our proof: it is precisely the special case of the implication (a) \Rightarrow (g) when $r = p^*$.

Let \mathcal{A} be an operator ideal. Let K be an \mathcal{A} -compact set and let (x_n) be an \mathcal{A} -null sequence. If \mathcal{B} is a larger operator ideal than \mathcal{A} , i.e. $\mathcal{A} \subset \mathcal{B}$, then, by definitions, clearly, K is also \mathcal{B} -compact and (x_n) is \mathcal{B} -null. In [1, Proposition 4.7], it was proved that

$$\mathcal{K}_{(p,r)} = \mathcal{I}_{(p,1,r^*)}^{\text{sur}} \circ \mathcal{K},$$

where $\mathcal{I}_{(p,1,r^*)}$ is the operator ideal of $(p, 1, r^*)$ -integral operators (for the definition of these general integral operators, see [22, 19.1.1]). This equality enables us to extend characterizations (d) and (f) of (p, r) -null sequences of Theorem 3.1 to even more larger operator ideal than $\mathcal{K}_{(p,r)}$, namely to $\mathcal{I}_{(p,1,r^*)}^{\text{sur}}$.

Proposition 3.3. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (a) (x_n) is (p, r) -null,
- (b) (x_n) is null and $\mathcal{I}_{(p,1,r^*)}^{\text{sur}}$ -compact,
- (c) (x_n) is $\mathcal{I}_{(p,1,r^*)}^{\text{sur}}$ -null.

Proof. As was mentioned, $\mathcal{K}_{(p,r)} = \mathcal{I}_{(p,1,r^*)}^{\text{sur}} \circ \mathcal{K}$. Hence, using Propositions 2.2 and 2.1, we have

$$\mathbf{k}_{(p,r)} = \mathcal{K}_{(p,r)}(\mathbf{b}) = \mathcal{I}_{(p,1,r^*)}^{\text{sur}}(\mathcal{K}(\mathbf{b})) = \mathcal{I}_{(p,1,r^*)}^{\text{sur}}(\mathbf{k}).$$

This shows that relatively (p, r) -compact sets are exactly $\mathcal{I}_{(p,1,r^*)}^{\text{sur}}$ -compact sets. The claim now follows from Theorems 3.1 and 1.2. \square

Concerning the special case when $r = p^*$, i.e., $r^* = p$, by definition, the operator ideal of *right p -nuclear operators* $\mathcal{N}^p = \mathcal{N}_{(p,1,p)}$ (cf. [22, 18.1.1] and, e.g., [26, p. 140]). Also, let \mathcal{P}_p denote the operator ideal of *absolutely p -summing operators* (p -summing operators in [15]). It was noted in [1, p. 157] that $\mathcal{P}_p^{\text{dual}} = \mathcal{I}_{(p,1,p)}^{\text{sur}}$. Therefore we can spell out, from Theorem 3.1 and Proposition 3.3, the following omnibus characterization of p -null sequences.

Corollary 3.4. *Let $1 \leq p < \infty$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (a) (x_n) is p -null,
- (b) (x_n) is null and relatively p -compact,
- (c) (x_n) is null and \mathcal{N}^p -compact,
- (d) (x_n) is null and \mathcal{K}_p -compact,
- (e) (x_n) is null and $\mathcal{P}_p^{\text{dual}}$ -compact,
- (f) (x_n) is \mathcal{N}^p -null,
- (g) (x_n) is \mathcal{K}_p -null,
- (h) (x_n) is $\mathcal{P}_p^{\text{dual}}$ -null,
- (i) (x_n) is uniformly p -null.

4. UNCONDITIONALLY AND WEAKLY (p, r) -NULL SEQUENCES

4.1. Unconditional and weak (p, r) -compactnesses. The (uniformly) (p, r) -null sequences and (p, r) -compactness in a Banach space X are defined in terms of (p, r) -convex hulls using the space $\ell_p(X)$ of absolutely p -summable sequences in X . In general, (p, r) -convex hulls can be defined using the space $\ell_p^w(X)$ of weakly p -summable sequences in X . This is a pretty old idea, going back at least to the paper [8, p. 51] by Castillo and Sanchez in 1993. In [8], the (p, p^*) -convex hull of $(x_n) \in \ell_p^w(X)$ was considered under the name of p^* -convex hull of (x_n) . In 2002, Sinha and Karn [27] developed some of their theory of p -compactness in a more general context

of weak p -compactness. In [27], also the (p, p^*) -convex hull of $(x_n) \in \ell_p^w(X)$ was used but under the name of p -convex hull of $(x_n) \in \ell_p^w(X)$.

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. In the present Section 4, we shall assume that the definition of the (p, r) -convex hull $(p, r)\text{-conv}(x_n)$ (see Section 2.1) is extended to $(x_n) \in \ell_p^w(X)$. In this case, the operator $\Phi_{(x_n)} : \ell_r \rightarrow X$ is also well defined and

$$(p, r)\text{-conv}(x_n) = \Phi_{(x_n)}(B_{\ell_r}).$$

But $\Phi_{(x_n)}$ need not be a compact operator any more (see, e.g., Section 4.3).

“Between” absolutely p -summable sequences $\ell_p(X)$ and weakly p -summable sequences $\ell_p^w(X)$, there is the Banach space $\ell_p^u(X)$ of *unconditionally p -summable sequences* (see, e.g., [10, 8.2, 8.3]; we follow [5] in our terminology). The space $\ell_p^u(X)$ is defined as the (closed) subspace of $\ell_p^w(X)$, formed by the $(x_n) \in \ell_p^w(X)$ satisfying $(x_n) = \lim_{N \rightarrow \infty} (x_1, \dots, x_N, 0, 0, \dots)$ in $\ell_p^w(X)$. The space $\ell_p^u(X)$ was introduced and thoroughly studied by Fourie and Swart [16] in 1979. In particular, it follows from [16, Theorem 1.4] that $\Phi_{(x_n)}$ is compact whenever $(x_n) \in \ell_p^u(X)$. In fact, $\Phi_{(x_n)} : \ell_{p^*} \rightarrow X$ is compact if and only if $(x_n) \in \ell_p^u(X)$ (see [16, Theorem 1.4] or, e.g., [10, 8.2]).

It is rather easy to see that our approach in Sections 2 and 3 goes through if $\ell_p(X)$ is replaced with the larger space $\ell_p^u(X)$. Let us start by fixing the relevant terminology and notation.

We define *relatively unconditionally* (respectively, *weakly*) (p, r) -compact sets in X by replacing $\ell_p(X)$ with $\ell_p^u(X)$ (respectively, with $\ell_p^w(X)$) in the definition of relatively (p, r) -compact sets. The classes of corresponding sets in all Banach spaces are denoted, respectively, by $\mathbf{u}_{(p,r)}$ and $\mathbf{w}_{(p,r)}$. So that $\mathbf{k}_{(p,r)} \subset \mathbf{u}_{(p,r)} \subset \mathbf{w}_{(p,r)}$ and $\mathbf{u}_{(p,r)} \subset \mathbf{k}$.

A linear operator $T : Y \rightarrow X$ is *unconditionally* (respectively, *weakly*) (p, r) -compact if $T(B_Y)$ is a relatively unconditionally (respectively, weakly) (p, r) -compact subset of X . Let $\mathcal{U}_{(p,r)}$ and $\mathcal{W}_{(p,r)}$ denote the classes of all unconditionally and weakly (p, r) -compact operators acting between arbitrary Banach spaces, so that $\mathcal{K}_{(p,r)} \subset \mathcal{U}_{(p,r)} \subset \mathcal{W}_{(p,r)}$ and $\mathcal{U}_{(p,r)} \subset \mathcal{K}$. It is clear from the definitions that $\mathbf{u}_{(p,r)} = \mathcal{U}_{(p,r)}(\mathbf{b})$ and $\mathbf{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\mathbf{b})$. An easy straightforward verification, as in the case of $\mathcal{K}_{(p,r)}$ (cf. [1, Propositions 2.1 and 2.2]), shows that $\mathcal{U}_{(p,r)}$ and $\mathcal{W}_{(p,r)}$ are surjective operator ideals.

Note that $\mathcal{W}_{(p,p^*)} = \mathcal{W}_p$, the class of *weakly p -compact operators*, studied in [27]. Similarly, in all cases, we shall write “ p ” instead of “ (p, p^*) ”, and speak, for instance, about the operator ideal \mathcal{U}_p of unconditionally p -compact operators.

4.2. Unconditionally (p, r) -null sequences. We define *(uniformly) unconditionally (p, r) -null sequences* in X by replacing $\ell_p(X)$ with $\ell_p^u(X)$ in the corresponding definitions of (p, r) -null and uniformly (p, r) -null sequences. The definition of the weak versions of these concepts will be given in Section 4.3; it turns out to be unreasonably restrictive to define the weak versions just by replacing $\ell_p(X)$ with $\ell_p^w(X)$.

Let $(x_n) \in \ell_p^u(X)$. Then (see [16, Lemma 1.2]) $x_n = \delta_n y_n$ for some $(\delta_n) \in c_0$ and $(y_n) \in \ell_p^w(X)$. Since, clearly,

$$\Phi_{(x_n)} = \sum_{n=1}^{\infty} e_n \otimes x_n = \sum_{n=1}^{\infty} \delta_n e_n \otimes y_n$$

(where $e_n \in \ell_r^*$ are the unit vectors) and (as well known and easy to verify) $(e_n) \in B_{\ell_r^w(\ell_r^*)}$, we have, by the definition of (t, u, v) -nuclear operators [22, 18.1.1],

$$\Phi_{(x_n)} \in \mathcal{N}_{(\infty, p^*, r^*)}(\ell_r, X).$$

Similarly, as in Section 2.1, we get that

$$\mathcal{U}_{(p, r)} = \mathcal{N}_{(\infty, p^*, r^*)}^{\text{sur}}.$$

This implies that

$$\mathcal{U}_{(p, r)} = \mathcal{K} \circ \mathcal{U}_{(p, r)} \circ \mathcal{K}.$$

Indeed, as in the proof of Proposition 2.4, $\mathcal{N}_{(\infty, p^*, r^*)} = \overline{\mathcal{F}} \circ \mathcal{N}_{(\infty, p^*, r^*)} \circ \overline{\mathcal{F}}$, and therefore

$$\mathcal{U}_{(p, r)} = (\overline{\mathcal{F}} \circ \mathcal{N}_{(\infty, p^*, r^*)} \circ \overline{\mathcal{F}})^{\text{sur}} \subset \overline{\mathcal{F}}^{\text{sur}} \circ \mathcal{N}_{(\infty, p^*, r^*)}^{\text{sur}} \circ \overline{\mathcal{F}}^{\text{sur}} = \mathcal{K} \circ \mathcal{U}_{(p, r)} \circ \mathcal{K},$$

because $\overline{\mathcal{F}}^{\text{sur}} = \mathcal{K}$ (see, e.g., [22, 4.7.13]).

Further, similarly to Proposition 2.2, we have $\mathbf{u}_{(p, r)} = \mathcal{N}_{(\infty, p^*, r^*)}(\mathbf{b}) = \mathcal{U}_{(p, r)}(\mathbf{b})$, which implies (cf. Proposition 2.4 and its proof) that $\mathbf{u}_{(p, r)} = \mathcal{N}_{(\infty, p^*, r^*)}(\mathbf{k}) = \mathcal{U}_{(p, r)}(\mathbf{k})$. Using the above facts and proceeding as in the proof of Theorem 3.1, we come to the omnibus characterization of unconditionally (p, r) -null sequences.

Theorem 4.1. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (a) (x_n) is unconditionally (p, r) -null,
- (b) (x_n) is null and relatively unconditionally (p, r) -compact,
- (c) (x_n) is null and $\mathcal{N}_{(\infty, p^*, r^*)}$ -compact,
- (d) (x_n) is null and $\mathcal{U}_{(p, r)}$ -compact,
- (e) (x_n) is $\mathcal{N}_{(\infty, p^*, r^*)}$ -null,
- (f) (x_n) is $\mathcal{U}_{(p, r)}$ -null,
- (g) (x_n) is uniformly unconditionally (p, r) -null.

Proof. It is mostly the verbatim version of the proof of Theorem 3.1. Only the claim that (x_n) is null whenever (x_n) is unconditionally (p, r) -null (see the implication (a) \Rightarrow (b)) needs to be commented (also for an easy reference in Section 4.3 below).

So, let (x_n) be unconditionally (p, r) -null. Then, as in the proof of (a) \Rightarrow (b) in Theorem 3.1, for every $\varepsilon > 0$ there are $N \in \mathbb{N}$ and $(z_k) \in \ell_p^u(X)$, $\|(z_k)\|_p^w \leq \varepsilon$, such that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_r}$, for all $n \geq N$. Hence,

$$\|x_n\| = \sup_{x^* \in B_{X^*}} |x^*(x_n)| \leq \sup_{x^* \in B_{X^*}} \sum_{k=1}^{\infty} |a_k^n x^*(z_k)| \leq \|(a_k^n)_k\|_r \|(z_k)\|_p^w \leq \varepsilon,$$

for all $n \geq N$, and therefore $x_n \rightarrow 0$. \square

Recall (see [17, Theorem 2.5] or, e.g., [22, 18.3.2]) that $\mathcal{N}_{(\infty, p, p^*)}$ coincides with the operator ideal K_p of *classical* p -compact operators. Following Fourie and Swart [16] or Pietsch [22, 18.3.1 and 18.3.2], a linear operator $T : Y \rightarrow X$ is called *p-compact*, i.e., $T \in K_p(Y, X)$, if there exist $A \in \mathcal{K}(Y, \ell_p)$ and $B \in \mathcal{K}(\ell_p, X)$ such that $T = BA$. Remark (see [20] and [23]) that \mathcal{K}_p and K_p are notably different as operator ideals.

Since $\mathcal{U}_{p^*} = \mathcal{U}_{(p^*, p)} = \mathcal{N}_{(\infty, p, p^*)}^{\text{sur}}$, we get that $K_p^{\text{sur}} = \mathcal{U}_{p^*}$ as a description of the surjective hull of K_p .

Let us spell out, from Theorem 4.1, an omnibus characterization of unconditionally p -null (i.e., (p, p^*) -null) sequences.

Corollary 4.2. *Let $1 \leq p < \infty$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (a) (x_n) is unconditionally p -null,
- (b) (x_n) is null and relatively unconditionally p -compact,
- (c) (x_n) is null and K_{p^*} -compact,
- (d) (x_n) is null and \mathcal{U}_p -compact,
- (e) (x_n) is K_{p^*} -null,
- (f) (x_n) is \mathcal{U}_p -null,
- (g) (x_n) is uniformly unconditionally p -null.

4.3. Weakly (p, r) -null sequences and weakly \mathcal{A} -null sequences.

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$, as before. What about the weakly (p, r) -null sequences? It would be natural to expect that they would form a subclass of weakly null sequences, but not a subclass of null sequences as in the case of (p, r) -null sequences (which might be called also absolutely (p, r) -null sequences) or unconditionally (p, r) -null sequences. This means that we cannot employ the “verbatim” definition: replacing $\ell_p(X)$ with $\ell_p^w(X)$.

Indeed (see the proof of Theorem 4.1), such a “weakly” (p, r) -null sequence would always be a null sequence. And, for instance, looking at $X = \ell_{p^*}$, every null sequence (x_n) in X would be uniformly “weakly” (p, p^*) -null, because the unit vector basis (e_k) of X belongs to $B_{\ell_p^w(X)}$ and, since $\Phi_{(e_k)} = I_X$, we have $x_n = \Phi_{(e_k)} x_n \in \|x_n\| p\text{-conv}(e_k)$.

To motivate a definition for weakly (p, r) -null sequences, let us make the following observation from Theorem 3.1, yielding two more characterizations of (p, r) -null sequences.

Proposition 4.3. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (i) (x_n) is (p, r) -null,
- (ii) for every $\varepsilon > 0$ there exist $(z_k) \in \ell_p(X)$ and $N \in \mathbb{N}$ such that $\|x_n\| \leq \varepsilon$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$,
- (iii) there exists $(z_k) \in \ell_p(X)$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n\| \leq \varepsilon$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$.

Proof. The implication (i) \Rightarrow (ii) is clear from the proof of Theorem 3.1, the first part of (a) \Rightarrow (b).

From (ii), it is clear that $x_n \rightarrow 0$, and also (fixing, e.g., $\varepsilon = 1$ and looking at the proof of Theorem 3.1, the second part of (a) \Rightarrow (b)) that (x_n)

is relatively (p, r) -compact. By Theorem 3.1, $(b) \Rightarrow (a)$, (x_n) is (p, r) -null, meaning that $(ii) \Rightarrow (i)$. By Theorem 3.1, $(b) \Rightarrow (g)$, (x_n) is uniformly (p, r) -null. Hence, assuming that $\varepsilon \leq 1$, condition (iii) holds (similarly to the implication $(i) \Rightarrow (ii)$ above).

Finally, $(iii) \Rightarrow (ii)$ is more than obvious, and we saw above that $(ii) \Leftrightarrow (i)$. \square

Looking at Proposition 4.3, it seems to be natural to make the following definitions.

Let (x_n) be a sequence in a Banach space X . We call (x_n) *weakly (p, r) -null* if for every $x^* \in X^*$ and every $\varepsilon > 0$ there exist $(z_k) \in \ell_p^w(X)$ and $N \in \mathbb{N}$ such that $|x^*(x_n)| \leq \varepsilon$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$. We call (x_n) *uniformly weakly (p, r) -null* if there exists $(z_k) \in \ell_p^w(X)$ with the following property: for every $x^* \in X^*$ and every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x^*(x_n)| \leq \varepsilon$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$.

Let \mathcal{A} be an operator ideal. In the present context, it would be natural to complement the Carl–Stephani theory with the concepts of weakly \mathcal{A} -null sequences and weakly \mathcal{A} -compact sets as follows.

We call a sequence (x_n) in a Banach space X *weakly \mathcal{A} -null* if there exist a Banach space Y , a weakly null sequence (y_n) in Y , and $T \in \mathcal{A}(Y, X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$. We say that a subset K of X is *weakly \mathcal{A} -compact* if K is of type $\mathcal{A}(\mathbf{w})$, i.e., $K \in \mathcal{A}(\mathbf{w})(X)$. (Recall that \mathbf{w} denotes the class of all relatively weakly compact sets.)

Two basic facts in the Carl–Stephani theory [7] are that the classes of \mathcal{A} -null and \mathcal{A}^{sur} -null sequences coincide, and so also do \mathcal{A} -compact and \mathcal{A}^{sur} -compact sets. The “weak” versions of these results do not hold.

Indeed, let \mathcal{V} denote the operator ideal of *completely continuous* operators, i.e., of operators who take weakly null sequences to null sequences. Then $\mathcal{V}^{\text{sur}} = \mathcal{L}$ (see, e.g., [22, 4.7.13]). Consequently, the weakly \mathcal{V} -null sequences are (precisely, because null sequences are \mathcal{K} -null, hence \mathcal{V} -null) the null sequences, but the weakly \mathcal{V}^{sur} -null sequences are precisely the weakly null sequences. Similarly, the weakly \mathcal{V} -compact sets are precisely relatively compact:

$$\mathcal{V}(\mathbf{w}) = \mathcal{V}(\mathcal{W}(\mathbf{b})) = (\mathcal{V} \circ \mathcal{W})(\mathbf{b}) = \mathcal{K}(\mathbf{b}) = \mathbf{k}$$

(see Remark 2.3 for the equality $\mathbf{w} = \mathcal{W}(\mathbf{b})$ and, e.g., [22, 3.1.3] for the equality $\mathcal{V} \circ \mathcal{W} = \mathcal{K}$). But $\mathcal{V}^{\text{sur}}(\mathbf{w}) = \mathbf{w}$.

However, for our purposes, the following analogue of the Lassalle–Turco Theorem 1.2, characterizing *weakly \mathcal{A} -null* sequences, will be sufficient.

Proposition 4.4. *Let \mathcal{A} be an operator ideal and let (x_n) be a sequence in a Banach space X .*

- (a) *If (x_n) is weakly \mathcal{A} -null, then (x_n) is weakly null and weakly \mathcal{A} -compact.*
- (b) *If (x_n) is weakly null and weakly \mathcal{A} -compact, then (x_n) is weakly \mathcal{A}^{sur} -null.*

In particular, if \mathcal{A} is surjective, then (x_n) is weakly \mathcal{A} -null if and only if (x_n) is weakly null and weakly \mathcal{A} -compact.

Proof. (a) We have $x_n = Ty_n$ for some $T \in \mathcal{A}(Y, X)$ and weakly null sequence (y_n) in Y . Hence (x_n) is weakly null. Since (y_n) is relatively weakly compact in Y , (x_n) is weakly \mathcal{A} -compact.

(b) We know that $(x_n) \subset T(K)$ for some $T \in \mathcal{A}(Y, X)$ and weakly compact subset K of Y . We may and shall assume that $0 \in K$. Denote by \overline{T} the injective associate of T . Then $T = \overline{T}q$, where $q : Y \rightarrow Z := Y/\ker T$ is the quotient mapping, and $\overline{T} \in \mathcal{A}^{\text{sur}}(Z, X)$ (by the definition of \mathcal{A}^{sur}).

If $q(K)$ and $\overline{T}(q(K)) = T(K)$ are endowed with their weak topologies from Z and X , respectively, then $\overline{T} : q(K) \rightarrow T(K)$ is a continuous bijection, hence a homeomorphism. Let $x_n = Tk_n = \overline{T}qk_n$ for some $k_n \in K$ and let $z_n = qk_n$. Then $z_n = \overline{T}^{-1}x_n \rightarrow \overline{T}^{-1}(0) = 0$ weakly (recall that $0 \in K$ and (x_n) is weakly null by the assumption). Since $x_n = \overline{T}z_n$ for all $n \in \mathbb{N}$, (x_n) is weakly \mathcal{A}^{sur} -null. \square

We saw (in Sections 2.2, 2.3, 4.1, 4.2) that $\mathbf{k}_{(p,r)} = \mathcal{K}_{(p,r)}(\mathbf{b}) = \mathcal{K}_{(p,r)}(\mathbf{k})$ and, similarly, $\mathbf{u}_{(p,r)} = \mathcal{U}_{(p,r)}(\mathbf{b}) = \mathcal{U}_{(p,r)}(\mathbf{k})$. Also $\mathbf{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\mathbf{b})$ (see Section 4.1). In general, $\mathcal{W}_{(p,r)}(\mathbf{b}) \neq \mathcal{W}_{(p,r)}(\mathbf{k})$. Indeed, as was mentioned in the beginning of Section 4.3, for $X = \ell_{p^*}$, one has $\Phi_{(e_k)} = I_X$. Hence, $\mathcal{W}_p(X, X) = \mathcal{L}(X, X)$ and therefore $\mathcal{W}_p(\mathbf{b})(X) = \mathbf{b}(X)$, but $\mathcal{W}_p(\mathbf{k})(X) = \mathbf{k}(X)$. We shall need the fact that in many cases $\mathcal{W}_{(p,r)}(\mathbf{b}) = \mathcal{W}_{(p,r)}(\mathbf{w})$.

Proposition 4.5. *Let $1 \leq p < \infty$ and $1 < r \leq p^*$ with $r < \infty$ if $p = 1$. Then*

$$\mathcal{W}_{(p,r)} = \mathcal{W}_{(p,r)} \circ \mathcal{W} \text{ and } \mathbf{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\mathbf{w}).$$

Proof. Let X and Y be Banach spaces and $T \in \mathcal{W}_{(p,r)}(Y, X)$. As in the case of \mathcal{W}_p in [27, pp. 20–21] and of $\mathcal{K}_{(p,r)}$ (see Section 2.1), we get a natural factorization $T = \overline{\Phi}_{(x_n)}S$ with $(x_n) \in \ell_p^w(X)$, where $\overline{\Phi}_{(x_n)}$ is the injective associate of $\Phi_{(x_n)}$ and $S \in \mathcal{L}(Y, Z)$, where $Z := \ell_r/\ker \Phi_{(x_n)}$. Since $\Phi_{(x_n)} \in \mathcal{W}_{(p,r)}(\ell_r, X)$, we have $\overline{\Phi}_{(x_n)} \in \mathcal{W}_{(p,r)}^{\text{sur}}(Z, X) = \mathcal{W}_{(p,r)}(Z, X)$, because $\mathcal{W}_{(p,r)}$ is surjective. Since ℓ_r is reflexive, also Z is, and therefore $S \in \mathcal{W}(Y, Z)$. This proves that $\mathcal{W}_{(p,r)} = \mathcal{W}_{(p,r)} \circ \mathcal{W}$. Now, using this, we have

$$\mathbf{w}_{(p,r)} = \mathcal{W}_{(p,r)}(\mathbf{b}) = (\mathcal{W}_{(p,r)} \circ \mathcal{W})(\mathbf{b}) = \mathcal{W}_{(p,r)}(\mathcal{W}(\mathbf{b})) = \mathcal{W}_{(p,r)}(\mathbf{w}). \quad \square$$

Remark 4.6. We do not know whether Proposition 4.5 holds in the “limit” case $r = 1$, i.e., for $\mathcal{W}_{(p,1)}$. It does not hold in the other “limit” case $p = 1$, $r = \infty$, i.e., for $\mathcal{W}_1 = \mathcal{W}_{(1,\infty)}$. Indeed, as we saw above, $\mathcal{W}_1(c_0, c_0) = \mathcal{L}(c_0, c_0)$, and hence

$$\mathbf{w}_1(c_0) = \mathcal{W}_1(\mathbf{b})(c_0) = \mathbf{b}(c_0) \neq \mathbf{w}(c_0) = \mathcal{W}_1(\mathbf{w}).$$

In particular, $\mathcal{W}_{(1,\infty)} \not\subset \mathcal{W}$. In all other cases $\mathcal{W}_{(p,r)} \subset \mathcal{W}$. For $r \neq 1$, this is clear from Proposition 4.5. But $\mathcal{W}_{(p,1)} \subset \mathcal{W}_{(p,r)}$ (by the definition of $\mathcal{W}_{(p,\cdot)}$, because $B_{\ell_1} \subset B_{\ell_r}$).

Remark 4.7. In the case $p = 1$, $1 \leq r \leq p^*$, including also the case $p = 1$, $r = \infty$ (cf. Remark 4.6), Proposition 4.5 holds in a strong form for a large class of Banach spaces X . Namely, for X that does not contain c_0 isomorphically. In this case (and only in this case), $\ell_1^w(X) = \ell_1^u(X)$, by the

classical Bessaga–Pełczyński theorem [4, Theorem 5] (see, e.g., [10, 8.3]). Therefore (see Section 4.2),

$$\mathcal{W}_{(1,r)}(Y, X) = \mathcal{U}_{(1,r)}(Y, X) = (\mathcal{K} \circ \mathcal{U}_{(1,r)} \circ \mathcal{K})(Y, X)$$

for all Banach spaces Y , and

$$\mathbf{w}_{(1,r)}(X) = \mathbf{u}_{(1,r)}(X) = \mathcal{U}_{(1,r)}(\mathbf{k})(X) = \mathcal{N}_{(\infty, \infty, r^*)}(\mathbf{k})(X).$$

Keeping in mind that the operator ideal $\mathcal{W}_{(p,r)}$ is surjective (see Section 4.1) we come to an omnibus characterization of weakly (p, r) -null sequences.

Theorem 4.8. *Let $1 \leq p < \infty$ and $1 < r \leq p^*$ with $r < \infty$ if $p = 1$. For a sequence (x_n) in a Banach space X the following statements are equivalent:*

- (a) (x_n) is weakly (p, r) -null,
- (b) (x_n) is weakly null and relatively weakly (p, r) -compact,
- (c) (x_n) is weakly null and weakly $\mathcal{W}_{(p,r)}$ -compact,
- (d) (x_n) is weakly $\mathcal{W}_{(p,r)}$ -null,
- (e) (x_n) is uniformly weakly (p, r) -null.

Proof. (a) \Rightarrow (b) It is clear from the definition that $x_n \rightarrow 0$ weakly. Also, by the definition, we have (fixing, e.g., $\varepsilon = 1$) $N \in \mathbb{N}$ and $(z_k) \in \ell_p^w(X)$ such that $\{x_N, x_{N+1}, \dots\} \subset (p, r)\text{-conv}(z_k)$. Continuing verbatim to the proof of Theorem 3.1, the second part of (a) \Rightarrow (b), we see that (x_n) is relatively weakly (p, r) -compact.

Implications (b) \Leftrightarrow (c) and (c) \Leftrightarrow (d) are immediate from Propositions 4.5 and 4.4, respectively.

To prove that (d) \Rightarrow (e), let (x_n) be a weakly $\mathcal{W}_{(p,r)}$ -null sequence. Then there are a weakly null sequence (y_n) in a Banach space Y and an operator $T \in \mathcal{W}_{(p,r)}(Y, X)$ such that $x_n = Ty_n$ for all $n \in \mathbb{N}$. The weak (p, r) -compactness of T gives us a sequence $(w_k) \in \ell_p^w(X)$ such that $T(B_Y) \subset (p, r)\text{-conv}(w_k)$. We also have an $M > 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Now $(z_k) := (Mw_k) \in \ell_p(X)$ and $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \in \mathbb{N}$. As (x_n) is weakly null in X , for every $x^* \in X^*$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x^*(x_n)| \leq \varepsilon$ for all $n \geq N$. Hence, (x_n) is uniformly weakly (p, r) -null.

The implication (e) \Rightarrow (a) is clear from the definitions. \square

Remark 4.9. As we saw, all implications of Theorem 4.8, except (b) \Rightarrow (c), also hold in the “limit” cases $r = 1$ and $p = 1, r = \infty$. In the proof, we used that the implication (b) \Rightarrow (c) is immediate from Proposition 4.5 (see also Remark 4.6). We do not know whether Theorem 4.8 holds in these cases. If $p = 1$ and $1 \leq r \leq p^*$, Theorem 4.8 holds in a stronger form for those Banach spaces X that do not contain c_0 isomorphically. Indeed, by Remark 4.7, in condition (b), “weakly $(1, r)$ -compact” is the same as “unconditionally $(1, r)$ -compact” and in condition (c) “weakly $\mathcal{W}_{(1,r)}$ -compact” is the same as “ $\mathcal{U}_{(1,r)}$ -compact” and also the same as “ $\mathcal{N}_{(\infty, \infty, r^*)}$ -compact”. In condition (d), “weakly $\mathcal{W}_{(1,r)}$ -null” is the same as “weakly $\mathcal{U}_{(1,r)} \circ \mathcal{K}$ -null”, which is the same as “ $\mathcal{U}_{(1,r)}$ -null”, since compact operators take weakly null sequences to null sequences, i.e., $\mathcal{K} \subset \mathcal{V}$ (see, e.g., [22, 1.11.4]). This shows that in the special case when $p = 1, 1 \leq r \leq p^*$, and X does not contain c_0 isomorphically, all conditions of Theorem 4.1 are equivalent to the conditions of Theorem 4.8.

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KATI AIN, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY,
J. LIIVI 2, 50409 TARTU, ESTONIA
E-mail address: `kati.ain@ut.ee`

EVE OJA, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, TARTU UNIVERSITY,
J. LIIVI 2, 50409 TARTU, ESTONIA; ESTONIAN ACADEMY OF SCIENCES, KOHTU 6, 10130
TALLINN, ESTONIA
E-mail address: `eve.oja@ut.ee`